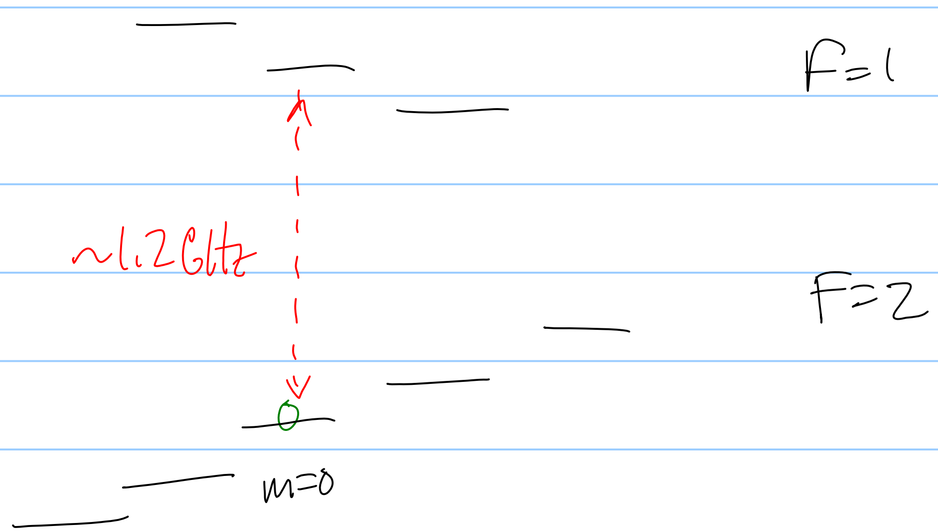
Quantum two-level systems and clocks

PHYS514

Note Title 2/15/2005

Before we do the solution, let's look at a real example:

$^9Be^+$ hyperfine states:



What if the atoms start in $|2,0\rangle$?

* First example of a selection rule in action!

What states can we connect to in F=1?

Assume that there is a large static field along z

$$H_I = -\vec{\mu}_e \cdot \vec{B} - \vec{\mu}_N \cdot \vec{B}$$
$$= 2\mu_0 \vec{S} \cdot \vec{B} - g_N \mu_N \vec{I} \cdot \vec{B}$$
$$= 2\mu_0 \left(\frac{1}{\sqrt{2}} S_+ B_-^{(1)} - \frac{1}{\sqrt{2}} S_- B_+^{(1)} + S_z B_z^{(1)} \right) - g_N \mu_N \cdot \left(\frac{1}{\sqrt{2}} I_+ B_-^{(1)} - \frac{1}{\sqrt{2}} I_- B_+^{(1)} + I_z B_z^{(1)} \right)$$

*see notes on spherical tensors

with the spherical basis as: $\hat{e}_+ = \frac{-1}{\sqrt{2}}(\hat{x} + i\hat{y})$, $\hat{e}_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$

Given: $\vec{B}^{(1)}(t) = \left(B_0^{(1)} \hat{e}_0 - B_+^{(1)} \hat{e}_- - B_-^{(1)} \hat{e}_+ \right) \cos(\omega t)$

It's not easy to see how this will work out. In general, you must calculate Rabi rates exactly by solving the time-dependent Schrödinger equation! See Alicia Kollar's notes for hydrogen, for example.

Note the following: $|F, m_F\rangle = \alpha |\underbrace{m_F + 1/2}_{m_I}, \underbrace{-1/2}_{m_S}\rangle + \beta |m_F - 1/2, 1/2\rangle$

If you compute $\langle F', m_F' | H_I | F, m_F \rangle$, you will find that ...

$B_0^{(1)}$ drives $\Delta m_F = 0$: $|2,0\rangle \longleftrightarrow |1,0\rangle$

$B_\pm^{(1)}$ drive $\Delta m_F = \pm 1$: $|2,0\rangle \longleftrightarrow |1,-1\rangle$

$|2,0\rangle \longleftrightarrow |1,1\rangle$

check the notes on spherical tensors to understand this $\longrightarrow$ this is correct if increasing m_F increases the energy (i.e., $\gamma < 0$, like an electron)

So, using these "selection rules", we can restrict the Hilbert space and make our complicated atom a 2-level system.

On to (or back to) the quantum case

$$H = -\vec{\mu}\cdot\vec{B} \qquad \vec{B} = B_0\hat{z} + B_1\cos(\omega t+\varphi)\hat{x}$$

$$H = -\mu\sigma_z B_0 - \mu\sigma_x B_1\cos(\omega t+\varphi)$$

$$= -\mu\sigma_z B_0 - \mu B_1 \frac{1}{2}(\sigma_+ + \sigma_-)\cdot\frac{1}{2}\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]$$

$$= -\mu\sigma_z B_0 + \Omega\left(\sigma_+ + \sigma_-\right)\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]$$

$$H = \hbar\begin{pmatrix} \frac{\omega_0}{2} & \Omega\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right] \\ \Omega\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right] & -\omega_0/2 \end{pmatrix}$$

$$\Omega = \frac{-\mu B_1}{4\hbar} \qquad \omega_0 = \frac{-2\mu B_0}{\hbar}$$

"Rabi frequency"

To solve using the rotating frame:

Schrodinger's equation:

$$\psi = a_+|+\rangle + a_-|-\rangle$$

① $$i\dot{a}_+ = \omega_0 a_+(t) + \Omega\left[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right] a_-$$

② $$i\dot{a}_- = \Omega\left[e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right] a_+(t) - \omega_0 a_-(t)$$

Now, define new coefficients:

$$b_+(t) = e^{i\omega t/2} a_+(t)$$
$$b_-(t) = e^{-i\omega t/2} a_-(t)$$

These are the coefficients in the quantum "rotating frame"

Then, our coupled equations become:

∘ RWA

① $$i\frac{d}{dt}\left(e^{-i\omega t/2} b_+\right) = \frac{\omega_0}{2} e^{-i\omega t/2} b_+ + \Omega\left(e^{-i(\omega t+\phi)} + \cancel{e^{i(\omega t+\phi)}}\right) e^{i\omega t/2} b_-$$

$$\frac{\omega}{2} e^{-i\omega t/2} b_+ + i e^{-i\omega t/2} \dot{b}_+ = \frac{\omega_0}{2} e^{-i\omega t/2} b_+ + \Omega e^{-i\omega t/2} e^{-i\phi} b_-$$

$$i\dot{b}_+ = \frac{\omega_0 - \omega}{2} b_+ + \Omega e^{-i\varphi} b_-$$

① $$i\dot{b}_+ = -\frac{\delta}{2} b_+ + \Omega e^{-i\varphi} b_-$$

② $$i\dot{b}_- = \Omega e^{i\varphi} b_+ + \frac{\delta}{2} b_-$$

coupled equations, but not time-dependent!

$$|\tilde{\psi}\rangle = b_+(t)|+\rangle + b_-(t)|-\rangle$$

$$\tilde{H} = \hbar \begin{pmatrix} -\delta/2 & \Omega e^{-i\varphi} \\ \Omega e^{i\varphi} & \delta/2 \end{pmatrix}$$

just like our static problem!

General solution:

$$|1\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2} e^{i\varphi}|-\rangle$$

$$|2\rangle = -\sin\theta/2\, |+\rangle + \cos\theta/2\, e^{i\varphi}|-\rangle$$

eigenstates

where $\tan\theta = \frac{-2\Omega}{\delta}$

$E_1 = \frac{\hbar}{2}\sqrt{\delta^2 + 4\Omega^2}$ $\qquad$ $E_2 = -\frac{\hbar}{2}\sqrt{\delta^2 + 4\Omega^2}$

Rabi oscillations:

let's say that at $t=0$, $|\psi(0)\rangle = |+\rangle$

$$|\psi(0)\rangle' = \cos\frac{\theta}{2}|1\rangle - \sin\theta/_2 |2\rangle$$ in the rotating frame

this state time evolves as:

$$|\psi(t)\rangle' = \cos\theta/_2\, e^{-iE_1t/\hbar}|1\rangle - \sin\theta/_2\, e^{-iE_2t/\hbar}|2\rangle$$

and $P_- = |\langle -|\psi(t)\rangle|^2$

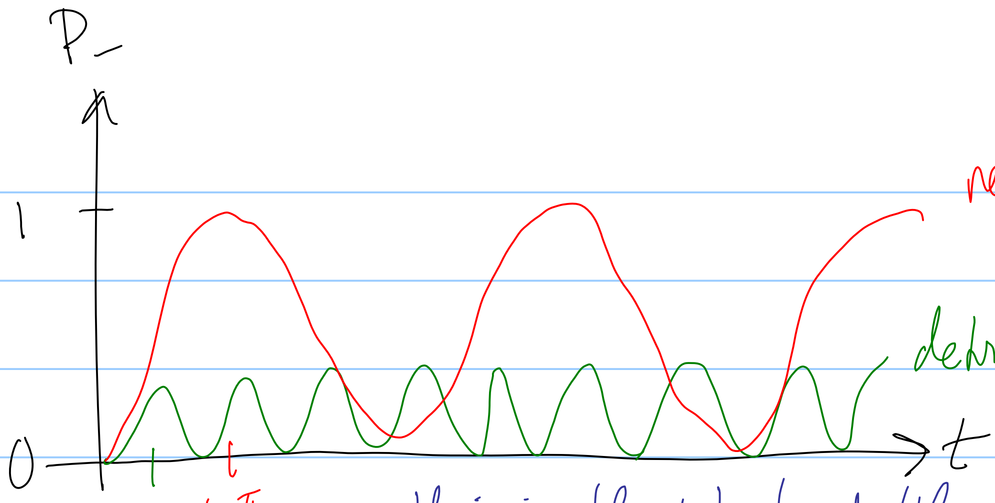
$$= \left|\cos\theta/_2 \sin\theta/_2 e^{i\varphi} e^{-iE_1t/\hbar} - \sin\theta/_2\, e^{-iE_2t/\hbar} \cos\theta/_2\, e^{i\varphi}\right|^2$$

⋮ math (note – no dependence on φ for this case)

$$\boxed{P_- = \frac{4\Omega^2}{4\Omega^2+\delta^2}\sin^2\left[\frac{t}{2}\sqrt{4\Omega^2+\delta^2}\right]}$$

as δ increases, size of oscillation decreases

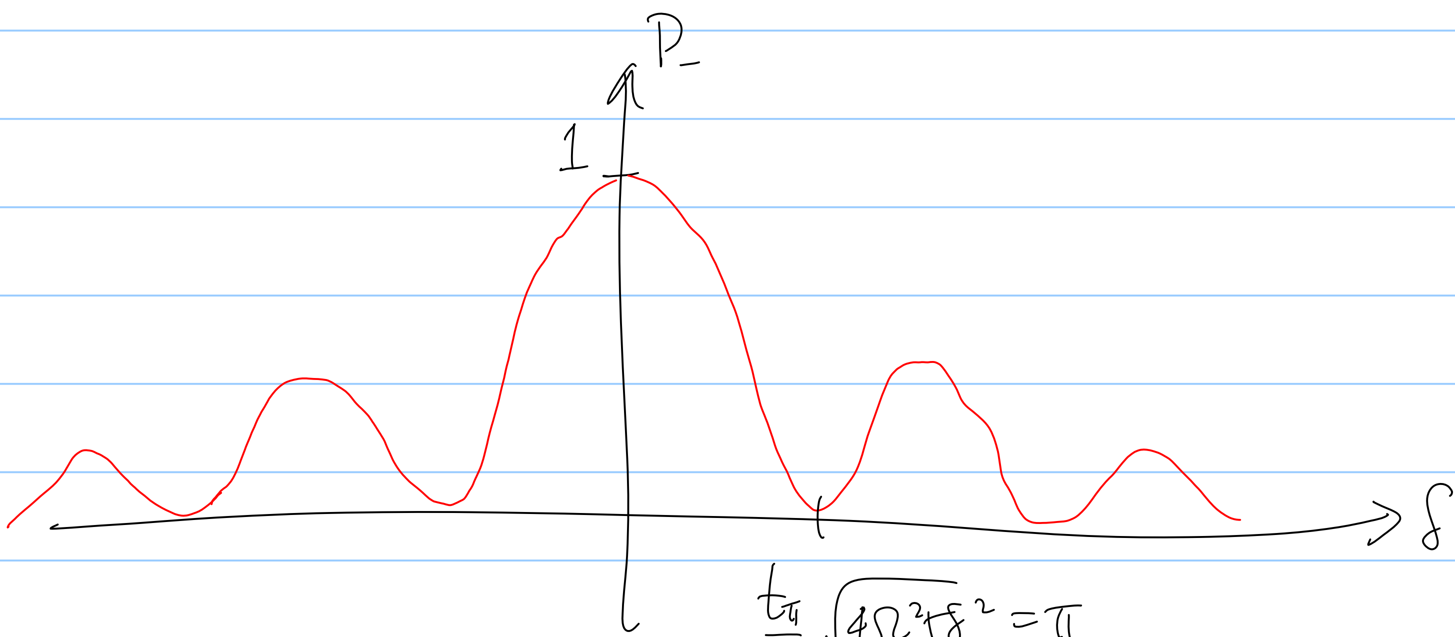
$\sin^2(\Omega t)$ for $\delta = 0$, faster "effective Rabi rate" as δ increases



$t = \pi/2\Omega$ ⟵ this is the Wineland / Ramsey standard — beware of factors of 2! (many people replace $\Omega \to \Omega/2$)

What does the lineshape look like?

i.e. set $t_\pi = \pi/2\Omega$, vary δ



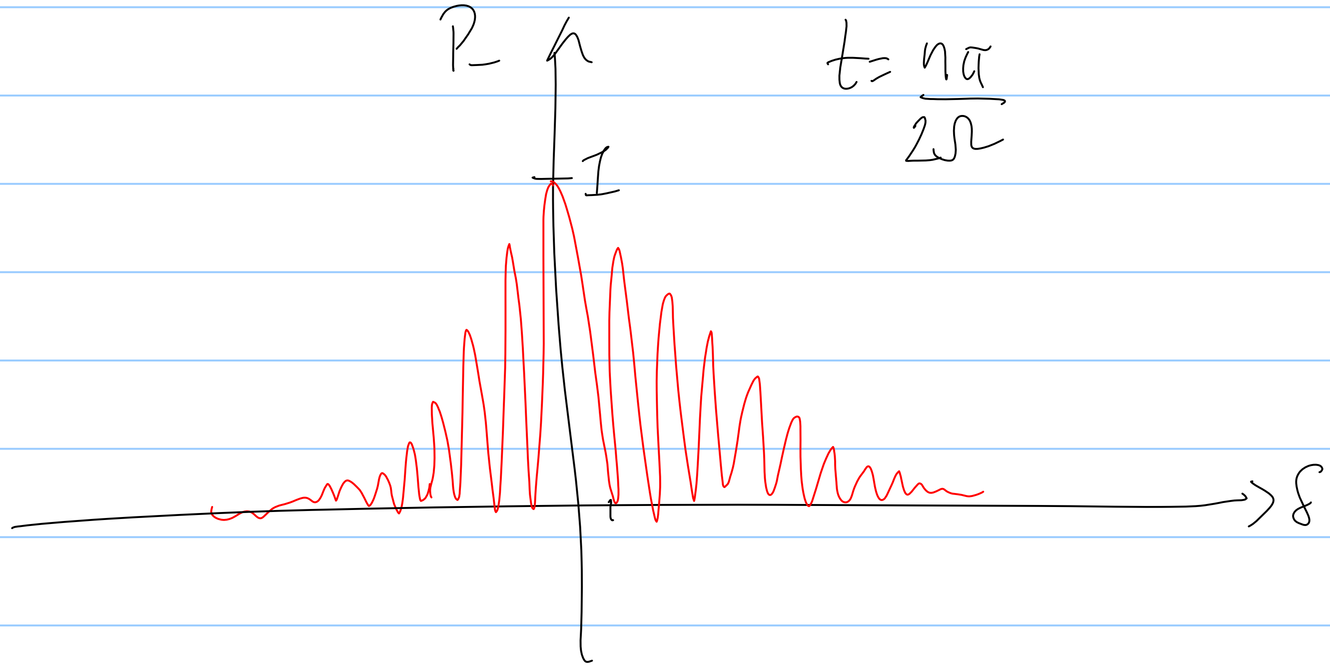
$$\frac{t_\pi}{2}\sqrt{4\Omega^2+\delta^2} = \pi$$

$$\frac{\pi}{2\Omega}\frac{1}{2}\sqrt{4\Omega^2+\delta^2} = \pi$$

$$\frac{1}{4\Omega}\sqrt{4\Omega^2+\delta^2} = 1$$

$$\delta = \Omega\, 2\sqrt{3}$$

for an n-π pulse :



$$t = \frac{n\pi}{2\Omega}$$

$\boxed{\text{An application has just presented itself}}$

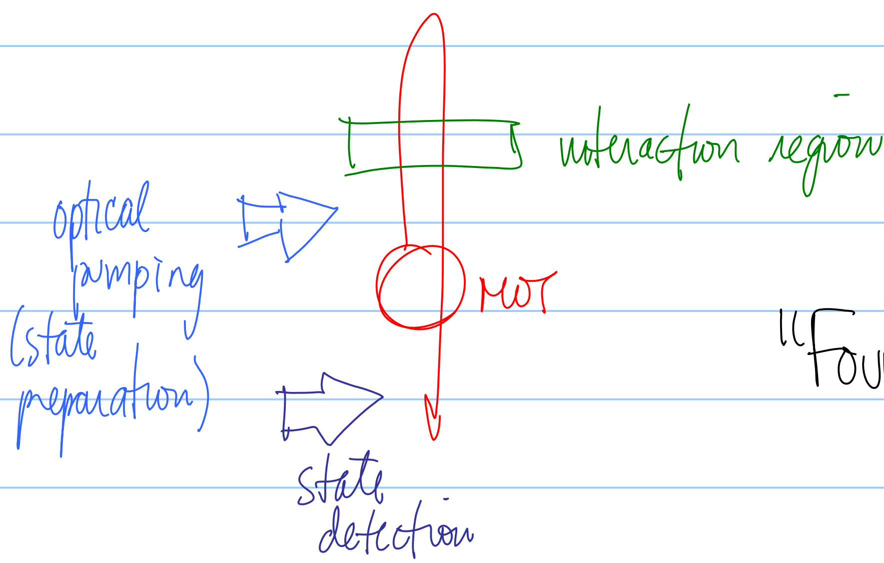
Measuring ω_0 — this is the goal of a clock experiment

Why not use an n-π pulse to go to arbitrary precision?

This can be hard!

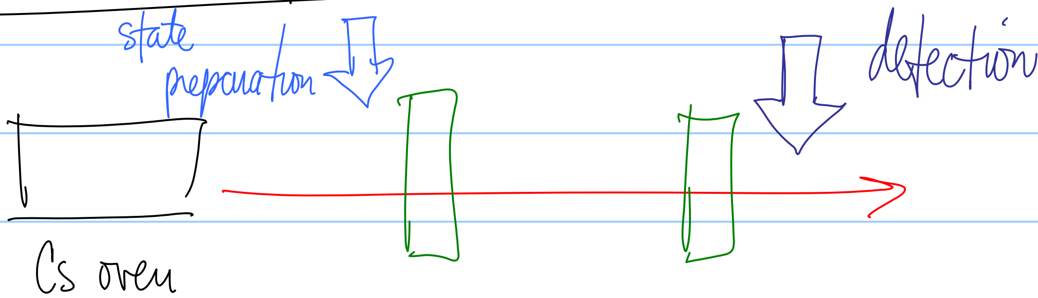
$$\Delta\omega = \frac{\Delta\omega_{FWHM}}{S/N}$$

$\boxed{\text{NIST-F1}}$ measures ^{133}Cs ΔE_{hfs} to 1.7 parts in 10^{15}



"Fountain" clock

original clock (up to NIST-7) used a beam:



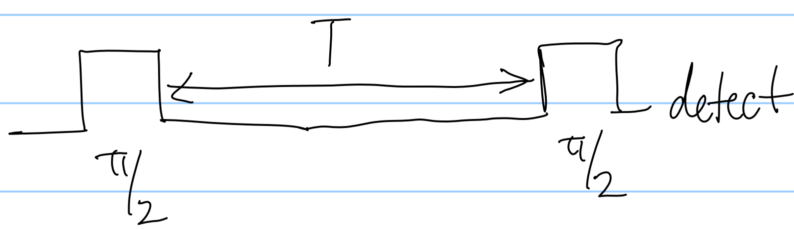
To measure ω_0 using n-π pulse is hard:

- cavity pulling
- maintaining drive φ and amplitude over long time

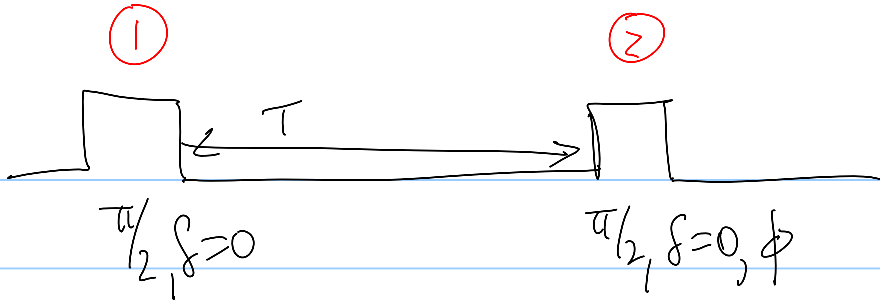
⋮

many other problems!

Ramsey's trick:



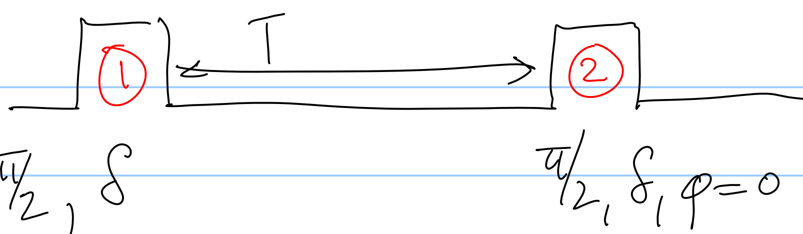
When the pulses are on-resonance ($\delta=0$), it's easy to see what happens:



* this type of experiment can be used to measure qubit coherence - more later!

- After 1, Bloch vector is along $-\hat{y}$ (assuming $\gamma < \delta$)
- During T, Bloch vector stays along $-\hat{y}$ in the rotating frame (it's precessing in the lab frame)
- At 2, if $\varphi = 0$, then spin is flipped
  - depending on φ, we should get full oscillations in P_- (or S_z)

What about off-resonance?



note that this is set on resonance!

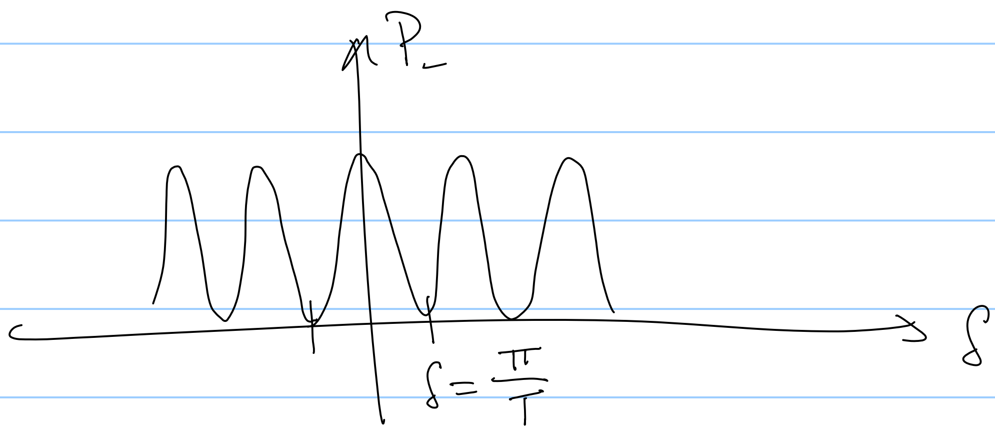
Imagine δ is small

- After 1, Bloch vector is almost along $-\hat{y}$

- During T, precesses at δ (keep the rotating frame going at ω)

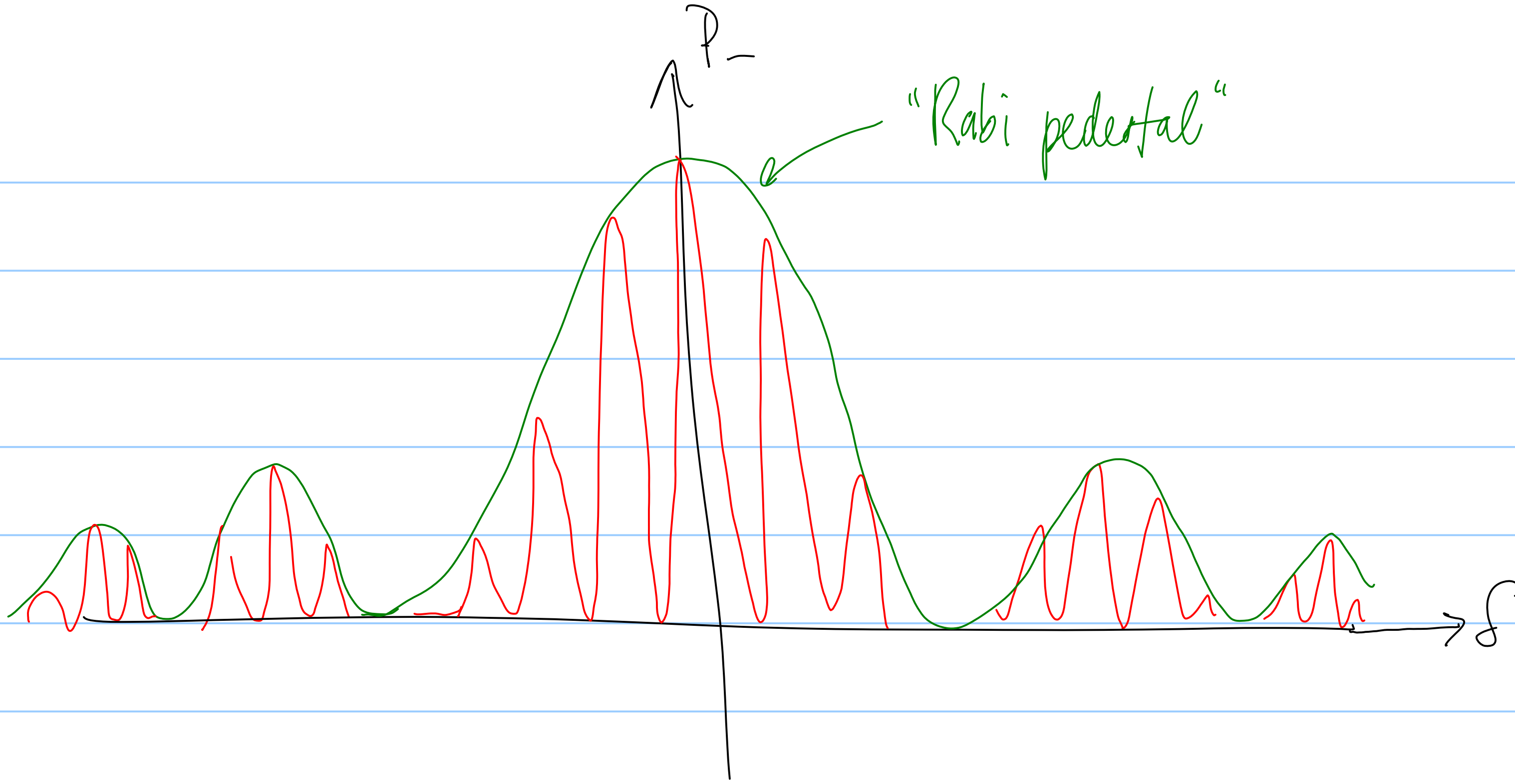
So, after T, the vector ends up almost in the plane at angle $\varphi = \delta T$

- After T, $P_- = \frac{1}{2} + \frac{1}{2}\cos(\delta T)$



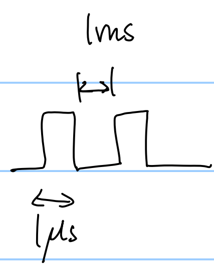
Exact result from Ramsey's book:

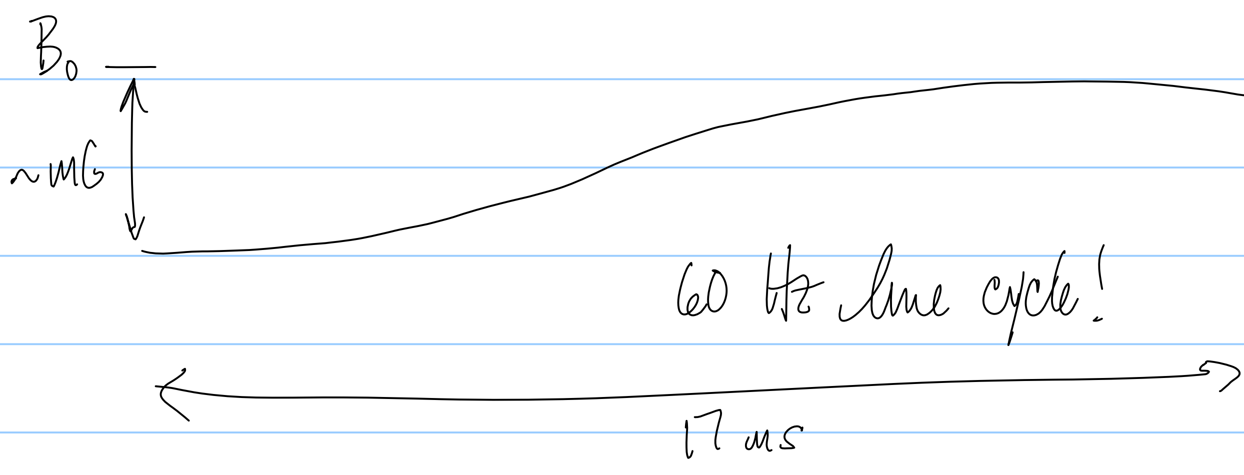
$$P_- = 4\,\frac{4\Omega^2}{4\Omega^2+\delta^2}\sin^2\left[\frac{t}{2}\sqrt{4\Omega^2+\delta^2}\right]\left\{\cos\left(\tfrac{1}{2}\delta T\right)\cos\left[\frac{t}{2}\sqrt{4\Omega^2+\delta^2}\right] - \frac{\delta}{\sqrt{4\Omega^2+\delta^2}}\sin\left(\tfrac{1}{2}\delta T\right)\sin\left[\frac{t}{2}\sqrt{4\Omega^2+\delta^2}\right]\right\}^2$$



There are all kinds of obnoxious effects when you try to do this in the lab

Trapped ions





So, B_0 roughly fixed over one run, but varies from shot-to-shot

What happens? Consider the resonant case, with fixed T and φ.

During T, we get different amounts of precession — small changes in ω_0 add up to a big effect if T is big!

Averaging then washes out the fringes (for each experiment, Bloch vector points somewhere differently in the plane before ②)

This is decoherence — we're not measuring everything
(more on this when we get to density matrices)

In a clock experiment, the fringes shift, which means that you measure ω_0 incorrectly!

An Appendix:

The following is a modern, non-density matrix solution to the two-level problem a lá Wineland:

$$H = H_0 + H_I$$

where $H_I = -\mu \sigma_x B_1 \cos(\omega t + \varphi)$

$$H_I = -\mu \frac{1}{2}(\sigma_+ + \sigma_-) \frac{1}{2}\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]$$

$$= \hbar \Omega (\sigma_+ + \sigma_-)\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]$$

$$\boxed{\text{where Rabi frequency } \Omega = -\frac{\mu B}{4\hbar}}$$

Now, go into the interaction picture, which is not the same as the rotating frame:

$$H_I' = e^{iH_0/\hbar\, t} H_I \, e^{-iH_0/\hbar\, t}$$

define $\delta = \omega - \omega_0$

$$H_I' = \begin{pmatrix} 0 & \hbar\Omega e^{-i(\delta t+\varphi)} \\ \hbar\Omega e^{i(\delta t+\varphi)} & 0 \end{pmatrix}$$

$\boxed{\text{with the RWA}}$

the Schrodinger equation has everything in it:

$$i\hbar \frac{\partial\psi}{\partial t} = H_I'\psi$$

$$\psi = c_1|+\rangle + c_2|-\rangle \qquad |c_1|^2 + |c_2|^2 = 1$$

<u>then:</u>

$$\left.\begin{aligned} \dot{c}_1 &= -i\, e^{-i(\delta t+\varphi)}\, c_2 \\ \dot{c}_2 &= -i\, e^{i(\delta t+\varphi)}\, c_1 \end{aligned}\right\}$$

these equations are solved using Laplace transforms

General solution:

$$\Psi(t) = \begin{bmatrix} e^{-i\frac{\Delta}{2}t}\left[\cos\left(\frac{X_{n',n}}{2}t\right) + i\frac{\Delta}{X_{n',n}}\sin\left(\frac{X_{n',n}}{2}t\right)\right] & -2i\frac{\Omega_{n',n}}{X_{n',n}}e^{-i\left(\frac{\Delta}{2}t-\phi-\frac{\pi}{2}|n'-n|\right)}\sin\left(\frac{X_{n',n}}{2}t\right) \\ -2i\frac{\Omega_{n',n}}{X_{n',n}}e^{+i\left(\frac{\Delta}{2}t-\phi-\frac{\pi}{2}|n'-n|\right)}\sin\left(\frac{X_{n',n}}{2}t\right) & e^{i\frac{\Delta}{2}t}\left[-i\frac{\Delta}{X_{n',n}}\sin\left(\frac{X_{n',n}}{2}t\right) + \cos\left(\frac{X_{n',n}}{2}t\right)\right] \end{bmatrix}\Psi(0)$$

$\Delta \to \delta$

$\Omega_{n',n} \to \Omega$

$X_{n',n} \to \sqrt{\delta^2 + 4\Omega^2}$